JOURNAL OF APPROXIMATION THEORY **42**, 14–26 (1984)

Strong Uniqueness in L^p Spaces

JAMES ANGELOS

Department of Mathematics, Central Michigan University, Mt. Pleasant, Michigan 48859, U.S.A.

AND

ALAN EGGER

Department of Mathematics, South Dakota School of Mines and Technology, Rapid City, South Dakota 57701, U.S.A.

Communicated by E. W. Cheney

Received October 20, 1982; revised August 17, 1983

Let V be a finite dimensional subspace of L^p, $1 < p < \infty$. For $f \in L^p \backslash V$, it is shown that the best approximation to f from V is strongly unique of order $\alpha = 2$ or p. Let V be an n-dimensional Haar subspace of $L^1[a, b]$, the continuous functions on $[a, b]$ with the L^1 norm. Let $f \in L^1[a, b] \backslash V$, that is Lipschitz and so that $V_1 = \text{span}\{V, f\}$ is a Haar subspace. Then it is shown that the best approximation to f from V is strongly unique of order 2.

0. INTRODUCTION

Given a Banach space X, a subset V, and an element $f \in X \backslash V$ such that f has a unique best approximation $g^* \in V$, we shall say that g^* is *strongly unique* at f if there exists a $\gamma = \gamma(f) > 0$ such that, for all $g \in V$,

$$\|f - g\| \geqslant \|f - g^*\| + \gamma \|g - g^*\|. \tag{0.1}$$

Similarly, we shall say g^* is *strongly unique of order* $\alpha (\alpha > 1)$ at f if, for some $M > 0$, there exists $\gamma = \gamma(f, M) > 0$ such that, for all $g \in V$ with $\|g^* - g\| \leqslant M$,

$$\|f - g\| \geqslant \|f - g^*\| + \gamma \|g - g^*\|^{\alpha}. \tag{0.2}$$

The concept of strong uniqueness has been extensively studied in the spaces $C(T)$ with the uniform norm, T a compact subset of $[a, b]$, and V a Haar (Chebyshev) subspace. This strong uniqueness property plays an

0021-9045/84 $3.00

important role in the Remes algorithm in this setting. It is known [5] that in smooth Banach spaces, in particular $L^p(T, \Sigma, \mu)$, $1 < p < \infty$, strong uniqueness will not in general hold.

Strong uniqueness of order α has been shown to hold in $C[a,b]$ for monotone approximation with $\alpha = 2$[3]. It can be easily shown that strong uniqueness of order $\alpha (\alpha < 1)$ is impossible and that strong uniqueness is strictly a local property. This can be seen by use of the following easily established fact.

LEMMA A. *Let $f \in X \backslash V$ and let $g^* \in V$ be the unique best approximation to f from V. Assume that* $\dim V < \infty$, *then g^* is strongly unique of order α, $\alpha \geqslant 1$ if and only if*

$$\underline{\lim}_{k \to \infty} \frac{\|f - g_k\| - \|f - g^*\|}{\|g_k - g^*\|^\alpha} > 0$$

for all sequences $\{g_k\}$ in $V \backslash \{g^*\}$ with $\lim_{k \to \infty} \|g_k - g^*\| = 0$.

In the following, we shall study strong uniqueness of order α in certain L^p spaces, $1 \leqslant p < \infty$.

1. Strong Uniqueness in $L^1[a,b]$

It is well known that best approximations need not exist in the general L^1 approximation problem. Even if a best approximation exists, it need not be unique. Interestingly, it is known that under fairly general conditions the set of functions which have a strongly unique best approximation is dense in L^1[1].

The following example shows that strong uniqueness need not hold in $L^1[a,b]$, with Lebesgue measure.

EXAMPLE 1. Let $a = -1$, $b = 1$, $f(x) = x$ and define V to be the subspace of constant functions on $[-1, 1]$. It is seen that $g^* = 0$ and if $\lambda \in V$ with $|\lambda| \leqslant 1$, we have

$$\|f - \lambda\|_1 = 1 + \lambda^2 = \|f\|_1 + \tfrac{1}{4} \|\lambda\|_1^2.$$

Thus in this case, strong uniqueness of order 2 holds. By applying Lemma A, it can be shown that the order 2 cannot be replaced with any smaller order. In the following we shall show that in the L^1 norm strong uniqueness of order 2 holds for a large class of problems.

Let V be an n-dimensional subspace of $C[a,b]$ with the L^1 norm. Let $f \in C[a,b] \backslash V$. Suppose that f is Lipschitz with constant k on $[a,b]$. Define $V_1 = \operatorname{span}\{V, f\}$. Assume that V and V_1 are Haar subspaces. Under these conditions f has a unique best approximation g^* from $V[2]$. By translating by f we may assume that $g^* \equiv 0$. We shall show that under these conditions strong uniqueness of order 2 holds at f.

LEMMA 1.1. *Under the above hypotheses f has precisely n interior zeros, $z_1, z_2, \ldots, z_n$ in $[a, b]$ at each of which f changes sign. Furthermore, if $N(z_i, \delta) \equiv \{x: |x - z_i| < \delta\}$ then there exists $\delta > 0$ and $\gamma > 0$ such that*

(a) $N(z_i, \delta) \subset [a, b]$, $i = 1, \ldots, n$,

(b) $N(z_i, \delta) \cap N(z_j, \delta) = \varnothing$ *if* $i \neq j$,

(c) *for each* $g \in V$, $\|g\|_1 = 1$, *there exists an* i, $1 \leqslant i \leqslant n$ *such that* $|g(x)| > \gamma$ *for* $x \in N(z_i, \delta)$.

Proof. Conditions (a) and (b) are easily satisfied by choosing δ sufficiently small. Condition (c) follows from the compactness of the unit ball in V, since V is a Haar subspace.

LEMMA 1.2. *If $g \in V$ then*

$$\int_a^b |f(x) - g(x)|\, d\mu \geqslant \int_a^b |f(x)|\, d\mu + \int_S |g(x)|\, d\mu$$

where $S = \{x \in [a, b]\colon\ \sigma(g(x)) = \sigma(f(x))$ *and* $|g(x)| > 2|f(x)|\}$ *and* $\sigma(g(x)) = \operatorname{sign}(g(x))$.

Proof. Set $S_1 = \{x \in [a, b]\colon \sigma(f(x)) \neq \sigma(g(x))\}$, $S_2 = \{x \in [a, b]\colon \sigma(f(x)) = \sigma(g(x))\}$ and $S_3 = \{x \in [a, b]\colon \sigma(f(x)) = \sigma(g(x))$ and $|g(x)| > |f(x)|\}$. Then

$$\int_a^b |f(x) - g(x)|\, d\mu = \int_{S_1} (|f(x)| + |g(x)|)\, d\mu + \int_{S_2} (|f(x)| - |g(x)|)\, d\mu + 2\int_{S_3} (|g(x)| - |f(x)|)\, d\mu.$$

Since $g^* \equiv 0$, we also have that $\int_a^b \sigma(f(x))\, g(x)\, d\mu = 0$ for all $g \in V[2]$. Combining the two equations above we have that

$$\int_a^b |f(x) - g(x)|\, d\mu = \int_a^b |f(x)|\, d\mu + 2\int_{S_3} (|g(x)| - |f(x)|)\, d\mu$$

and so

$$\int_a^b |f(x) - g(x)|\, d\mu \geqslant \int_a^b |f(x)|\, d\mu + \int_S |g(x)|\, d\mu$$

as desired.

LEMMA 1.3. *There exists a $c > 0$ and $\beta > 0$ such that if $0 \leqslant \lambda \leqslant c$, then for all $g \in V$, $\|g\|_1 = 1$,*

$$\int_a^b |f(x) - \lambda g(x)|\, d\mu \geqslant \int_a^b |f(x)|\, d\mu + \beta\lambda^2.$$

Proof. By Lemma 1.2, we have

$$\int_a^b |f(x) - \lambda g(x)|\, d\mu \geqslant \int_a^b |f(x)|\, d\mu + \int_{S_\lambda} |\lambda g(x)|\, d\mu$$

where $S_\lambda = \{x \in [a, b]\colon \sigma(f(x)) = \sigma(g(x))$ and $|\lambda g(x)| > 2\,|f(x)|\}$. Since f is Lipschitz, there exists a positive constant k such that $|f(x) - f(y)| \leqslant k\,|k - y|$ for all x, $y \in [a, b]$, $x \neq y$. With δ and γ as in Lemma 1.1, choose $c = 2k\delta/\gamma$. Then $\mu\{x \in [a, b]\colon \sigma(g(x)) = \sigma(f(x))$, $\lambda\,|g(x)| > 2\,|f(x)|$ and $|g(x)| > \gamma\} > \gamma\lambda/2k$ for each $g \in V$, $\|g\|_1 = 1$. Thus for $0 < \lambda \leqslant c$, $\int_a^b |f - \lambda g(x)|\, d\mu \geqslant \int_a^b |f(x)|\, d\mu + \beta\lambda^2$ with $\beta = \gamma^2/2k$.

THEOREM 1.4. *Under the above hypotheses, if $M > 0$ is given, then there exists $\beta' > 0$ such that*

$$\|f - g\|_1 \geqslant \|f\| + \beta' \|g\|_1^2 \tag{1.1}$$

for all $g \in V$ satisfying $\|g\|_1 \leqslant M$.

Proof. For fixed M and the constant c from Lemma 1.3, inequality (1.1) holds for $g \in V$, $\|g\|_1 \leqslant c$, with constant β. By compactness, inequality (1.1) must also hold in the region $g \in V$, $M \geqslant \|g\|_1 \geqslant c$ with some constant β_c. Choose $\beta' = \min(\beta, \beta_c)$.

By translating the above problem, we may write Theorem 1.4 as follows.

THEOREM 1.5. *Let $[a, b]$ be a real interval and let V be an n-dimensional subspace of $C[a, b]$. Let $f \in C[a, b] \backslash V$ and suppose f is Lipschitz on $[a, b]$. Assume further that both V and* span$\{f, V\}$ *are Haar subspaces. Let g^* be the best approximation from V to f in the L^1 norm with Lebesgue measure. Then strong uniqueness of order 2 holds at f, i.e., there exist $\gamma = \gamma(M, f) > 0$ such that*

$$\|f - g\|_1 \geqslant \|f - g^*\|_1 + \gamma \|g - g^*\|_1^2$$

for all $g \in V$ satisfying $\|g\|_1 \leqslant M$.

Remark 1. Under the conditions of Theorem 1.5 strong uniqueness of order 2 holds. This need not be the lowest α for which strong uniqueness of order α holds. If $[a, b]$ is the interval $[-1, 1]$, V the subspace of constants, and $f(x)=x^3$, the hypotheses of Theorem 1.5 hold. The best approximation to f is $g^* \equiv 0$, and strong uniqueness holds of order 4/3 here.

Remark 2. The Lipschitz condition on f in Theorem 1.5 is necessary. To show this we again let $[a, b]=[-1, 1]$, and V be the subspace of constants. If $f(x)=x^{1/3}$ then the orther hypotheses of Theorem 1.5 hold but 4 is the lowest order for which strong uniqueness holds in this case.

2. Strong Uniqueness in L^p, $2 \leqslant p < \infty$

Throughout the next section we assume that (T, Σ, μ) is a positive measure space, $2 \leqslant p < \infty$, and that V is a nontrivial finite dimensional subspace of $L^p \equiv L^p(T, \Sigma, \mu)$. If $1 < p < \infty$ and $f \in L^p \backslash V$ then there exists a unique best approximation $g^* \in V$ to f. We shall need the following well-known result.

Theorem 2.1. (Characterization theorem). *Let $f \in L^p \backslash V$, $1 < p < \infty$, then $g^* \in V$ is the best approximation to f if and only if*

$$\int_T |f-g^*|^{p-1} \sigma(f-g^*) h \, d\mu = 0,$$

for all $h \in V$, where $\sigma(f-g^*) = \operatorname{sign}(f-g^*)$.

In the case $p=2$, a direct computation yields the following lemma.

Lemma 2.2. *If $p=2$, strong uniqueness of order 2 holds at f.*

Proof. Since $\|f-g\|_2^2 = \|f-g^*\|_2^2 + \|g-g^*\|_2^2$, it suffices to show that there exists $\gamma=\gamma(M, f) > 0$ such that for $\|g\|_2 \leqslant M$,

$$\sqrt{\|f-g^*\|_2^2 + \|g-g^*\|_2^2} \geqslant \|f-g^*\|_2 + \gamma \|g-g^*\|_2^2.$$

This is equivalent to finding $\gamma > 0$ such that $1 \geqslant 2\gamma \|f-g^*\|_2 + \gamma^2 \|g-g^*\|_2^2$ for $g \neq g^*$. $\|f-g^*\|_2$ is fixed, so for any fixed M, $\|g\|_2 \leqslant M$ implies $\|g-g^*\|_2 \leqslant \|g^*\|_2 + M$, so that such a γ always exists.

To obtain similar results for L^p, $2 < p < \infty$, we shall require the following lemmas.

Lemma 2.3. *If $p \geqslant 1$, $q > 0$ then there exists $M > 0$ and $\gamma > 0$ such that if $M \geqslant b/a \geqslant 0$ then $(a+b)^{1/p} \geqslant a^{1/p} + \gamma b$.*

Proof. It suffices to show that for sufficiently small positive x, $(1+x)^{1/p} \geqslant 1+\gamma x$. This follows since $\phi(x)=(1+x)-(1+\gamma x)^p$ is a nonnegative, increasing function of x in some neighborhood of 0.

Given functions h and g in L^p, we define $\operatorname{supp}(g) \equiv \{x\colon g(x) \neq 0\}$ and we shall say that h and g are *disjointly supported* if $\mu(\operatorname{supp}(g) \cap \operatorname{supp}(h)) = 0$. Otherwise, we shall say that they are *mutually supported.*

LEMMA 2.4. *Let $1 < p < \infty$. Let $h \in L^p$ satisfy $h \not\equiv 0$. Then there exists $M > 0$ and $\gamma > 0$ such that if $g \in L^p$, h and g are disjointly supported, and $\|g\|_p^p \leqslant M$ then*

$$\left(\int_T |h+g|^p \, d\mu\right)^{1/p} \geqslant \left(\int_T |h|^p \, d\mu\right)^{1/p} + \gamma \int_T |g|^p \, d\mu.$$

Proof. We have that

$$\left(\int_T |h+g|^p \, d\mu\right)^{1/p} = \left(\int_T |h|^p + |g|^p \, d\mu\right)^{1/p}.$$

By Lemma 2.3 there exists $M > 0$ and $\gamma > 0$ such that if $\int_T |g|^p \, d\mu \leqslant M$

$$\left(\int_T |h+g|^p \, d\mu\right)^{1/p} \geqslant \left(\int_T |f|^p \, d\mu\right)^{1/p} + \gamma \int_T |g|^p \, d\mu.$$

LEMMA 2.5. *If $w \in L^p$, $2 < p < \infty$, then $\|h\|_w = (\int_T |w|^{p-2} |h|^2 \, d\mu)^{1/2}$ is a seminorm on L^p.*

Proof. It suffices to show that if $g \in L^p$ then $\int_T |w|^{p-2} |g|^2 \, d\mu < \infty$. Observe that $|w|^{p-2} \in L^{p/(p-2)}$. Let $p' = p/(p-2)$, then if $1/p' + 1/q' = 1$, we have $q' = p/2$. Now $|g|^2 \in L^{p/2}$, so by Hölders inequality we have

$$\int_T |w|^{p-2} |g|^2 \, d\mu \leqslant \left(\int_T (|w|^{p-2})^{p/p-2} \, d\mu\right)^{(p-2)/p} \left(\int_T (|g|^2)^{p/2}\right)^{2/p} < \infty.$$

Hence $\|\cdot\|_w$ is a seminorm on L^p. In fact, $\|\cdot\|_w$ is a norm on any subspace which contains no nonzero element supported disjointly from w.

LEMMA 2.6. *Let $p > 2$, $f \in L^p \backslash V$, and g^* be the unique best approximation from V to f. If $f - g^*$ and g are mutually supported for each $g \not\equiv 0$ in V, then g^* is a best weighted L^2 approximation to f with weight function $|f - g^*|^{p-2}$.*

Proof. $\int_T |f - g^*|^{p-1} \sigma(f - g^*) h \, d\mu = 0$ for all $h \in V$ by Theorem 2.1. Hence $\int_T |f - g^*|^{p-2} (f - g^*) \, h \, d\mu = 0$ for all $h \in V$. Theorem 2.1 then yields that g^* is the unique best weighted L^2 approximation to f with weight function $|f - g^*|^{p-2}$.

THEOREM 2.7. *If $p > 2$ and $f - g^*$ and g are mutually supported for each $g \not\equiv 0$ in V then strong uniqueness of order 2 holds at f.*

Proof. By Lemma 2.2 and Lemma 2.6 we have that for each M there exists $\gamma > 0$ such that

$$\left(\int_T |f - g^*|^{p-2}(f - g)^2 \, d\mu\right)^{1/2} \geqslant \left(\int_T |f - g^*|^p \, d\mu\right)^{1/2}$$

$$+ \gamma \int_T |f - g^*|^{p-2} |g - g^*|^2 \, d\mu$$

for $\|g\|_w \leqslant M$. By an application of Holders inequality as in Lemma 2.5 and by dividing through by $(\int_T |f - g^*|^p \, d\mu)^{(p-2)/2p}$ we have the desired result for $\|g\|_w \leqslant M$. By the equivalency of $\|\cdot\|_p$ and $\|\cdot\|_w$ on the finite dimensional subspace V we have the desired result for $\|g\|_p \leqslant M_1$, where $M_1 > 0$.

We now derive a strong uniqueness result for the case when V contains functions which are disjointly supported from $f - g^*$. Let $p > 2$, $f \in L^p \backslash V$, and assume that $g^* \equiv 0$, where g^* is the best approximation from V to f. Let $V_1 = \{g \in V: \text{supp}(g) \subset S^c\}$ where $S = \text{supp}(f)$ and S^c denotes the complement of S in T. Now, V_1 is a subspace of V and we may decompose V into a direct sum $V = V_1 \oplus V_2$, where $V_2 \subset V$ and $V_1 \cap V_2 = \{0\}$. Hence, if $g \in V_2$ and $g \equiv 0$ on S, then $g \equiv 0$. Each $g \in V$ may be written uniquely in the form $g = g_1 + g_2$ with $g_1 \in V_1$ and $g_2 \in V_2$. Hence

$$\int_T |f - g|^p \, d\mu = \int_S |f - g_2|^p \, d\mu + \int_{S^c} |g_1 + g_2|^p \, d\mu.$$

Since $0 \in V_2$ is the best approximation from V_2 to f, there exists γ_1 and M_1 such that if $\|g\|_p < M_1$

$$\int_T |f - g|^p \, d\mu \geqslant (\|f\|_p + \gamma_1 \| g_2\|_p^2)^p + \int_T |g_1 + g_2|^p \, d\mu - \int_S |g_2|^p \, d\mu.$$

By Lemma 2.3 if $\|g\|_p \leqslant N$ there exists $\gamma_2 > 0$ such that

$$\|f - g\|_p \geqslant \|f\|_p + \gamma_1 \| g_2\|_p^2 + \gamma_2\left(\int_T |g_1 + g_2|^p \, d\mu - \int_S |g_2|^p \, d\mu\right).$$

Hence

$$\|f - g\|_p \geqslant \|f\|_p + \gamma_2 \| g_1 + g_2\|_p^p + \gamma_1 \| g_2\|_p^2 - \gamma_2 \|g_2\|_p^p.$$

Now $p > 2$, so that there exists $M_2 > 0$ such that for $\|g_2\|_p \leqslant M_2$, $\gamma_1 \| g_2\|_p^2 > \gamma_2 \| g_2\|_p^p$. Thus, for $\| g\|_p \leqslant M_2$, we have $\|f - g\|_p \geqslant \|f\|_p + \gamma_2 \| g_1 + g_2\|_p^p$.

Theorem 2.8. *If $p \geqslant 2$ and V is a finite dimensional subspace of L^p with $f \in L^p \backslash V$, then strong uniqueness of order p holds at f.*

Remark. The proof of Theorem 2.7 actually yields the result that there exists β_1, β_2 both positive such that if $\|g\|_p \leqslant M$, $\|f-g\|_p \geqslant \|f-g\|_p + \beta_1 \|g-g^*\|_p^p + \beta_2 \|(g-g^*)\chi_S\|_p^2$ where

$$\chi_S(x) = \begin{cases} 0 & \text{if } x \notin \operatorname{supp}(f-g^*), \\ 1 & \text{if } x \in \operatorname{supp}(f-g^*). \end{cases}$$

Thus, in the case that $f-g^*$ is mutually supported with each nonzero g in V, we have that strong uniqueness of order 2 holds. The mutual support condition will frequently be satisfied and holds, for example, when $\operatorname{span}(V, f)$ is Haar. Note that in the case that there exists a nonzero function $g \in V$ such that $f-g^*$ and g have disjoint support Lemma A implies that p is the smallest order for which strong uniqueness can hold at f.

Remark. A more general approach can be used in the case where X is a sufficiently smooth Banach space, i.e., its norm is at least twice Frechet differentiable on the subspace V and is positive definite on $S(V) = \{g \in V: \|g\| = 1\}$; $\dim V < \infty$. In this case, by the use of Taylor's theorem on the norm, g^* is strongly unique of order 2. This order is also the best possible.

For the L^p space, $2 \leqslant p < \infty$, in the mutual support situation the norm will satisfy the above differentiability conditions. For a nice treatment of norm differentiation see [4], which includes the L^p norms.

3. STRONG UNIQUENESS IN L^p, $1 < p < 2$

Let $1 < p < 2$ and (T, Σ, μ) be a positive measure space. Let V be an n-dimensional subspace of $L^p \equiv L^p(T, \Sigma, \mu)$. Suppose $f \in L^p \backslash V$ and 0 is the best approximation from V to f. We shall show that strong uniqueness of order 2 holds in this case. If $g \in V$ we may write.

$$\int_T |f-g|^p \, d\mu = \int_{Z(f)} |g|^p \, d\mu + \int_{S_1} |(|f| + |g|)^p \, d\mu + \int_{S_2} (|f| - |g|)^p \, d\mu + \int_{S_3} (|g| - |f|)^p \, d\mu$$

where $Z(f) = \{t: f(t) = 0\}$; $S = T \backslash Z(f)$

$$S_1 = S \cap \{t: \sigma(f(t) \neq \sigma(g(t))\},$$

$$S_2 = S \cap \{t: \sigma(f(t)) = \sigma(g(t)) \text{ and } |g| < |f|\},$$

$$S_3 = S \cap \{t: \sigma(f(t) = \sigma(g(t)) \text{ and } |f| < |g|\}.$$

Now using the Taylor expansion $(a+t)^p = a^p + pa^{p-1}t + (p(p-1)/2)(a+\psi t)^{p-2}t^2$ for $a > 0$ and some ψ, $0 \leqslant \psi \leqslant 1$ we have that there exist functions $\theta_1, \theta_2, \theta_3, 0 \leqslant \theta_i \leqslant 1, i = 1, 2, 3$ for which

$$\int_T |f-g|^p\,d\mu$$

$$= \int_{Z(f)} |g|^p\,d\mu + \int_{S_1} (|f|^p + p\,|f|^{p-1}\,|g| + \tfrac{1}{2}p(p-1)(|f| + \theta_1\,|g|)^{p-2}g^2)\,d\mu$$

$$+ \int_{S_2} (|f|^p - p\,|f|^{p-1}|g| + \tfrac{1}{2}p(p-1)(|f| - \theta_2\,|g|)^{p-2}g^2)\,d\mu$$

$$+ \int_{S_3} (|g|^p - p\,|g|^{p-1}\,|f| + \tfrac{1}{2}p(p-1)(|g| - \theta_3\,|f|)^{p-2}f^2)\,d\mu. \qquad (3.1)$$

Now on S_3, we have $0 < |f| < |g|$, so $|g|^{p-1}\,|f| < |f|^{p-1}\,|g|$ and $-p\,|g|^{p-1}\,|f| > -p\,|f|^{p-1}\,|g|$; thus

$$\int_T |f-g|^p\,d\mu \geqslant \int_{Z(f)} |g|^p\,d\mu + \int_T |f|^p\,d\mu - p\int_T g\,|f|^{p-1}\sigma(f)\,d\mu$$

$$+ \tfrac{1}{2}p(p-1)\int_{S_1} (|f| + \theta_1\,|g|)^{p-2}\,g^2\,d\mu$$

$$+ \int_{S_2} (|f| - \theta_2|g|)^{p-2}g^2\,d\mu + \int_{S_3} (|g| - \theta_3\,|f|)^{p-2}f^2\,d\mu).$$

By Theorem 2.1, the third integral on the right of the above inequality is zero and since the first and final integrals are nonnegative we have that

$$\int_T |f-g|\,d\mu \geqslant \int_T |f|^p\,d\mu + \tfrac{1}{2}p(p-1)\left(\int_{S_1} (|f| + \theta_1\,|g|)^{p-2}g^2\,d\mu\right.$$

$$\left. + \int_{S_2} (|f| - \theta_2\,|g|)^{p-2}g^2\,d\mu\right).$$

Define $\theta(x)$ by $\theta(x) \equiv \theta_1(x)$ on S_1 and $\theta(x) \equiv -\theta_2(x)$ on S_2. Then

$$\int_T |f-g|^p\,d\mu \geqslant \int_T |f|^p\,d\mu + \tfrac{1}{2}p(p-1)\int_{U_g} (|f| + \theta\,|g|)^{p-2}\,|g|^2\,d\mu \qquad (3.2)$$

where $U_g = \{x \in \operatorname{supp}(f) : |g(x)| \leqslant |f(x)|\}$ and $|\theta| \leqslant 1$ on U_g. We shall now consider three cases.

Case 1. Suppose $\{g_i\}_{i=1}^n$ is a basis for V and $g_1, \ldots, g_n$ are linearly independent on $S = \text{supp}(f)$. Let $A_k = \{x: |f(x)| \geqslant 1/k\}$ and let $B_{k,m} = \{x \in A_k : |g_i(x)| \leqslant m, \text{ for all } i\}$.

LEMMA 3.1. *There exists k and m positive integers such that $g_1,..., g_n$ are linearly independent on $B_{k,m}$.*

Proof. Assume otherwise. Then for each fixed k and for each m we may select $\alpha_1^m,..., \alpha_n^m$ such that $\sum_{i=1}^n |\alpha_i^m| = 1$ and $\sum_{i=1}^n \alpha_i^m g_i = 0$ a.e. on $B_{k,m}$. Thus some subsequence of vectors $(\alpha_1^m,..., \alpha_n^m)$ converges to $(\alpha_1,..., \alpha_n)$ with $\sum_{i=1}^n |\alpha_i| = 1$. Let $W_0 = \{x \in A_k : |g_i(x)| = \infty$ for some $i\}$ and let $W_m = \{x \in B_{k,m} : \sum_{i=1}^n \alpha_i^m g_i(x) \neq 0\}$. Then $\mu(W_0) = 0$ and $\mu(W_m) = 0$, hence $W = \bigcup_0^\infty W_m$ has measure zero. If $x \in A_k \backslash W$, then $x \in B_{k,m} \backslash (W_0 \cup W_m)$ for large m, hence $\sum_{i=1}^n \alpha_i g_i = 0$ a.e. on A_k. Thus $g_1,..., g_n$ are linearly dependent on A_k for each k. Hence for each k we may select $\beta_1^k,..., \beta_n^k$ such that $\sum_{i=1}^n |\beta_i^k| = 1$ and $\sum_{i=1}^n \beta_i^k g_i = 0$ a.e. on A_k. Again some subsequence of $(\beta_1^k,..., \beta_n^k)$ converges to $(\beta_1,..., \beta_n)$ such that $\sum_{i=1}^n |\beta_i| = 1$.

Let

$$V_0 = \{x \in \text{supp}(f) : |g_i(x)| = \infty \text{ for some } i\},$$

$$V_k = \left\{x \in A_k : \sum_{i=1}^n \beta_i^k g_i(x) \neq 0\right\}, \qquad k = 1, 2,...,$$

and

$$V = \bigcup_0^\infty V_k.$$

Then $\mu(V_k) = \mu(V_0) = \mu(V) = 0$ for each $k = 1, 2,...$. For $x \in \text{supp}(f) \backslash V$ we have $x \in A_k \backslash V$ for all large k and $\sum_{i=1}^n \beta_i^k g_i = 0$ for all k, thus $\sum_{i=1}^n \beta_i g_i = 0$ a.e. on $\text{supp}(f)$. This contradiction establishes the lemma.

Select k and m as in Lemma 3.1. For any $g = \sum_{i=1}^n \gamma_i g_i$ we have that $\|g\|_* = \sum_{i=1}^n |\gamma_i|$ is a norm on V and hence there are positive numbers a and b such that $a \|\cdot\|_p \leqslant \|\cdot\|_* \leqslant b \|\cdot\|_p$ on V. Suppose $\|g\|_p \leqslant 1/kmb$, then, for $x \in B_{k,m}$, $|g(x)| = |\sum \gamma_i g_i(x)| \leqslant m \|g\|_* \leqslant \|g\|_p mb$ and $\|g\|_p mb < |f(x)|$ implying $x \in U_g$. Thus, $B_{k,m} \subset U_g$. Furthermore, $0 < |f(x)| + \theta |g(x)| \leqslant 2|f(x)|$ and since $1 < p < 2$,

$$(|f(x)| + \theta |g(x)|)^{p-2} \geqslant 2^{p-2} |f(x)|^{p-2}.$$

Now U_g is measurable and $(|f| + \theta |g|)^{p-2} |g|^2$ is integrable on U_g. Thus, $|f|^{p-2} |g|^2$ is integrable on $B_{k,m}$ and

$$\int_{B_{k,m}} |f|^{p-2} |g|^2 \, d\mu \leqslant \int_{U_g} (|f| + \theta |g|)^{p-2} g^2 \, d\mu.$$

From Lemma 2.5 we have that $\| g\|_{**} = (\int_{B_{k,m}} |f|^{p-2} g^2\, d\mu)^{1/2}$ is a norm on V since $|f|^{p-2} > 0$ on $B_{k,m}$ and $g_1,..., g_n$ are linearly independent there. Thus, by the equivalence of norms in finite dimensional spaces, we have that there exists $\gamma_1 > 0$ such that if $\| g\|_p \leqslant 1/mbk$

$$\int_T |f-g|^p\, d\mu \geqslant \int_T |f|^p + \gamma_1 \left(\int_T |g|^p\, d\mu\right)^{2/p}$$

by (3.2) and (2.1). Hence, by Lemma 2.3 we have that for $\| g\|_p$ sufficiently small, there exists $\gamma > 0$ such that

$$\|f-g\|_p \geqslant \|f\|_p + \gamma \| g\|_p^2.$$

Therefore, strong uniqueness of order 2 at f holds in this case.

Case 2. Suppose $\mu(\operatorname{supp}(g) \cap \operatorname{supp}(f)) = 0$ for all $g \in V$. Then, as in Section 3, strong uniqueness of order p holds at f.

Case 3. Suppose that there exists a $g \in V$, $g \not\equiv 0$, such that $\mu(\operatorname{supp}(g) \cap \operatorname{supp}(f)) = 0$, but not all nonzero $g \in V$ satisfy this condition. This is, in fact, the true general case. As before break up V into $V_1 = \{ g \in V\colon \mu(\operatorname{supp}(g) \cap \operatorname{supp}(f)) = 0\}$ and V_2 the subspace such that $V = V_1 \oplus V_2$, i.e., if $h \in V_2$ and $h(x) = 0$ for all $x \in \operatorname{supp}(f)$ then $h \equiv 0$. Each $g \in V$ may be uniquely written in the form $g = g_1 + g_2$ where $g_1 \in V_1$ and $g_2 \in V_2$. Then we have that

$$\int_T |f-(g_1+g_2)|^p\, d\mu = \int_S |f-g_2|^p\, d\mu + \int_{Z(f)} |g_1+g_2|^p\, d\mu$$

where $S = \operatorname{supp}(f)$. By Lemma 2.3, we have that given $M_0 > 0$ there exists $\gamma_0 = \gamma_0(f, M_0) > 0$ such that $\| g\|_p \leqslant M_0$ implies

$$\|f-g\|_p \geqslant \left(\int_S |f-g_2|^p\, d\mu\right)^{1/p} + \gamma_0 \left(\int_{Z(f)} |g_1+g_2|^p\, d\mu\right).$$

By Case 1, 0 is the strongly unique best approximation of order 2 to f from V_2 on S. Thus, there exists $\gamma_1 > 0$ and $M_1 > 0$ such that if $(\int_S |g_2|^p\, d\mu)^{1/p} < M_1$,

$$\|f-g\|_p \geqslant \|f\|_p + \gamma_1 \left(\int_S |g_2|^p\, d\mu\right)^{2/p} + \gamma_0 \left(\int_{(Z(f))} |g_1+g_2|^p\, d\mu\right).$$

On V_2, the norms $\| \cdot \|_p$ and $\| g_2\|' = (\int_S |g|^p\, d\mu)^{1/p}$ are equivalent. Hence, for some $M_2 > 0$ and $\gamma_2 > 0$, we have

$$\|f-g\|_p \geqslant \|f\|_p + \gamma_2 \| g_2\|_p^2 + \gamma_0 \int_{Z(f)} |g_1+g_2|^p\, d\mu \tag{3.2}$$

provided $g \in V$ is such that $\| g_2\|_p \leqslant M_2$. Now since $V = V_1 \oplus V_2$ there exists $M_3 \geqslant 0$ such that $\| g\|_p \leqslant M_3$ implies $\| g_2\|_p \leqslant M_2$, hence (3.2) holds for all $g \in V$ with $\| g\|_p \leqslant M_3$. Set $\gamma_3 = \min(\gamma_0, \gamma_2)$. We consider the following two subcases.

Subcase (a). If $\| g_2\|_p \geqslant \frac{1}{4} \| g\|_p$, we have that

$$\|f - g\|_p \geqslant \|f\|_p + \frac{\gamma_3}{16} \| g\|_p^2.$$

Subcase (b). If $\| g_2\|_p < \frac{1}{4} \| g\|_p$, then

$$\|f - g\|_p \geqslant \|f\|_p + 3 \int_{Z(f)} | g|^p \, d\mu.$$

Thus $\|f - g\|_p \geqslant \|f\|_p + \gamma_3(\| g\|_p^p - \| g_2\|_p^p)$. Hence,

$$\|f - g\|_p \geqslant \|f\|_p + \frac{\gamma_3}{2^p} \| g\|_p^p.$$

So strong uniqueness of order 2 must hold in Case 3 since $1 < p < 2$.

Remark. In this case, the orders of strong uniqueness are not necessarily best possible, as the following example illustrates. Let V be the subspace of constant functions in $L^p[-2, 2]$ and define $f \in L^p[-2, 2]$ to be -1 on $[-2, -1]$, 1 on $[1, 2]$ and zero elsewhere. Then, $g^* \equiv 0$ and f and g are mutually supported for all $g \in V$, but g^* is strongly unique of order p.

4. Conclusion

In the previous sections, it was shown that in the L^p norms strong uniqueness of order 2 holds for a wide class of problems. For the case when $p \geqslant 2$, these orders are shown to be best possible. However. for the case $1 < p < 2$, these order are not necessarily best possible.

Acknowledgments

The authors wish to thank Professors Darrell Schmidt and Gerald Taylor for their aid and guidance. It should also be noted that much of the research for this report was completed while the first author was a student at Montana State University, Bozeman, Montana, and the second author, a student at Colorado State University, Ft. Collins, Colorado.

REFERENCES

1. J. R. ANGELOS AND D. P. SCHMIDT, Strong uniqueness in $L^1(X, \Sigma, \mu)$, *in* "Approximation Theory IV" (L. Schumaker, Ed.), Academic Press, New York, 1983, in press.
2. B. R. KRIPKE AND T. J. RIVLIN, Approximation in the metric of $L^1(X, \mu)$, *Trans. Amer. Math. Soc.* **119** (1965), 101–122.
3. D. SCHMIDT, Strong unicity and Lipschitz conditions of order 1/2 for monotone approximation, *J. Approx. Theory* **27** (1979), 19–33.
4. K. SUNDARESAN, Smooth banach spaces, *Math. Ann.*, **173** (1967), 191–199.
5. D. E. WULBERT, Uniqueness and differential characterization of approximations from manifolds of functions, *Amer. J. Math.* **18** (1971), 350–366.